# Some Clebsch-Gordan type linearisation relations and other polynomial expansions associated with a class of generalised multiple hypergeometric series arising in physical and quantum chemical applications



1988 J. Phys. A: Math. Gen. 21 4463

(http://iopscience.iop.org/0305-4470/21/23/026)

J. Phys. A: Math. Gen. **21** (1988) 4463-4470. Printed in the UK

**ADDENDUM**

# Some Clebsch–Gordan type linearisation relations and other polynomial expansions associated with a class of generalised multiple hypergeometric series arising in physical and quantum chemical applications

H M Srivastava

Department of Mathematics, University of Victoria, Victoria, British Columbia V8W 2Y2, Canada

Received 24 November 1987, in final form 10 June 1988

**Abstract.** In the present sequel to Srivastava's work, a number of new expansions (in series of various classes of hypergeometric polynomials) are derived for the general multivariable hypergeometric function which provides an interesting and useful unification of numerous families of hypergeometric functions of one, two or more variables encountered naturally in a wide variety of physical and quantum chemical applications. By suitably specialising some of these polynomial expansions, Clebsch–Gordan type linearisation relations are deduced for the products of several Jacobi or Laguerre polynomials. It is also shown how one of the linearisation relations involving Laguerre polynomials, presented here, would yield the corrected (and modified) versions of a couple of results given recently by Niukkanen.

## 1. Introduction

Hypergeometric series (and hypergeometric polynomials) in one and more variables arise naturally and rather frequently in a wide variety of problems in theoretical physics and applied mathematics, and indeed also in engineering sciences, statistics and operations research (see, for examples, Srivastava and Karlsson (1985, § 1.7) and the various references therein). In fact, a considerable field of physical and quantum mechanical situations (such as Schrödinger's wave mechanics) lead naturally to such hypergeometric polynomials as the Bessel polynomials $y_n(x, \alpha, \beta)$, and the classical orthogonal polynomials including, for example, the Hermite polynomials $H_n(x)$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and the Laguerre polynomials $L_n^{(\alpha)}(x)$, and also to such special cases of the Jacobi polynomials as the Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(x)$, the Legendre polynomials $P_n(x)$ and the Tchebycheff polynomials (of the first and second kinds) $T_n(x)$ and $U_n(x)$. Since

$$y_n(x, \alpha, \beta) = n!\left(-\frac{x}{\beta}\right)^n L_n^{(1-\alpha-2n)}\left(\frac{\beta}{x}\right) \tag{1}$$

and

$$H_{2n+\varepsilon}(x) = (-1)^n 2^{2n+\varepsilon} n!\, x^{\varepsilon} L_n^{(\varepsilon-1/2)}(x^2) \qquad \varepsilon = 0 \text{ or } 1 \tag{2}$$

all of the aforementioned orthogonal polynomials are easily recoverable from the classical Jacobi and Laguerre polynomials. The Jacobi and Laguerre polynomials also play a significant role in approximate variational solutions of complex many-electron systems; indeed, in such variational methods, the basis functions are quite frequently connected with these two classes of orthogonal polynomials (for example, rotator functions or the Wigner $D$ functions via the Jacobi polynomials, and hydrogen-like functions via the Laguerre polynomials).

For any two given sequences of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ and $\{q_n(x)\}_{n=0}^{\infty}$, encountered (for example) in quantum mechanical applications, it is often convenient (and, sometimes, necessary) to express the product $p_l(x)p_m(x)$ as a linear combination of the polynomials $p_n(x)$ or $q_n(x)$, that is, to make use of a linearisation relation of the Clebsch-Gordan type:

$$P_l(x)p_m(x)=\sum_n \lambda_n p_n(x) \tag{3}$$

or of the (*modified*) Clebsch-Gordan type:

$$p_l(x)p_m(x)=\sum_n \mu_n q_n(x). \tag{4}$$

Much more general linearisation relations than those characterised by (3) and (4) above (involving, for example, the products of three or more Jacobi or Laguerre polynomials) are becoming increasingly important in atomic and nuclear shell theories. In particular, the hydrogen-like functions (or, equivalently, the Laguerre polynomials) have been frequently encountered in recent years as perspective basis functions for variational calculations of molecular electron wavefunctions. With this point in view, Niukkanen (1985) developed a linearisation relation for the product

$$t^k L_{m_1}^{(\alpha_1)}(x_1 t)\ldots L_{m_n}^{(\alpha_n)}(x_n t) \tag{5}$$

and discussed a number of particular cases of practical interest. For non-negative integer values of the parameter $k$ (and this is how $k$ is implicitly constrained in Niukkanen's work (Niukkanen 1985, p 1401, line 3)), the additional factor $t^k$ in (5) seems to serve no purpose whatsoever, since

$$t^k=(-1)^k k!\,L_k^{(-k)}(t) \qquad k=0,1,2,\ldots \tag{6}$$

which incidentally follows readily from a familiar functional relationship (e.g. Szegö 1975, p 102, equation (5.2.1)) between Laguerre polynomials of orders $k$ and $-k$.

The main object of the present paper is to show how linearisation relations for polynomial products like those in (5), but with *unrestricted* $k$, would result rather systematically from a number of substantially more general expansion formulae (in series of various classes of generalised hypergeometric polynomials) for the general multivariable hypergeometric function:

$$F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} \equiv F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\left(\begin{matrix}\boldsymbol{a}_0:\ \boldsymbol{a}_1;\ldots;\boldsymbol{a}_n;\\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\ldots;\boldsymbol{b}_n;\end{matrix}\ x_1,\ldots,x_n\right)$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\boldsymbol{a}_0)_{m_1+\ldots+m_n}}{(\boldsymbol{b}_0)_{m_1+\ldots+m_n}}\prod_{j=1}^{n}\left\{\frac{(\boldsymbol{a}_j)_{m_j}}{(\boldsymbol{b}_j)_{m_j}}\frac{x_j^{m_j}}{m_j!}\right\} \tag{7}$$

studied recently by Niukkanen (1983, 1984) and Srivastava (1985a, b, 1987); here $(\lambda)_m$ denotes the Pochhammer symbol given by the $\Gamma$-function quotient

$$\frac{\Gamma(\lambda+m)}{\Gamma(\lambda)}$$

and, for (absolute) convergence of the multiple hypergeometric series in (7),

$$1+q_0+q_k-p_0-p_k \geqslant 0 \qquad k=1,\ldots,n \tag{8}$$

it being understood that the equality in (8) holds true provided that, in addition, we have *either*

$$p_0 > q_0 \qquad \text{and} \qquad |x_1|^{1/(p_0-q_0)}+\ldots+|x_n|^{1/(p_0-q_0)} < 1 \tag{9}$$

*or*

$$p_0 \leqslant q_0 \qquad \text{and} \qquad \max\{|x_1|,\ldots,|x_n|\} < 1 \tag{10}$$

and that, under certain parametric constraints, the multiple hypergeometric series in (7) would converge also when

$$x_k = \pm 1 \qquad k=1,\ldots,n \tag{11}$$

together, of course, with the equality in (8). (See, for example, Srivastava (1987) for a number of other useful notations, conventions and definitions which will be employed throughout this paper.)

## 2. Polynomial expansions and linearisation relations

For a (real or complex) parameter $\mu$, let us begin by introducing three classes of hypergeometric polynomials defined by

$$B_m(t) = {}_{2+p+s}F_{q+r}\left[\begin{matrix} -m, \lambda+m, \boldsymbol{a}-\mu, \boldsymbol{d}; \\ \boldsymbol{b}-\mu, \boldsymbol{c}; \end{matrix}\, t\right] \tag{12}$$

$$B_m^*(t) = {}_{1+p+s}F_{q+r}\left[\begin{matrix} -m, \boldsymbol{a}-\mu, \boldsymbol{d}; \\ \boldsymbol{b}-\mu, \boldsymbol{c}; \end{matrix}\, t\right] \tag{13}$$

and

$$B_m^{(\alpha)}(t) = {}_{2+p+s}F_{2+q+r}\left[\begin{matrix} -m, 1+\beta/(1-\alpha), \boldsymbol{a}-\mu, \boldsymbol{d}; \\ \beta/(1-\alpha), \beta-\alpha m+1, \boldsymbol{b}-\mu, \boldsymbol{c}; \end{matrix}\, t\right] \tag{14}$$

where $m$ is a non-negative integer and the other parameters are unrestricted (in general), and, by analogy with the abbreviations implied in the definition (7),

$$\boldsymbol{c} = (c^1,\ldots,c^r) \qquad \boldsymbol{d} = (d^1,\ldots,d^s) \tag{15}$$

so that $\boldsymbol{c}$ and $\boldsymbol{d}$ are vectors with dimensions $r$ and $s$, respectively (see, for generalised hypergeometric ${}_pF_q$ notation, Srivastava and Karlsson (1985, p 19, equation 1.2(23) and following). Then, from the work of Srivastava and Panda (1974, 1976) containing several general classes of polynomial expansions for multivariable functions defined by multiple series or multiple Mellin-Barnes type contour integrals, it is not difficult to derive the following expansions for the generalised multiple hypergeometric function defined by (7):

$$t^{\mu}F^{p+p_0:p_1;\dots;p_n}_{q+q_0:q_1;\dots;q_n}\begin{pmatrix}x_1t\\ \vdots\\ x_nt\end{pmatrix}$$

$$\equiv t^{\mu}F^{p+p_0:p_1;\dots;p_n}_{q+q_0:q_1;\dots;q_n}\left(\begin{matrix}\boldsymbol{a},\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\dots;\boldsymbol{a}_n;\\ \boldsymbol{b},\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\dots;\boldsymbol{b}_n;\end{matrix}\ x_1t,\dots,x_nt\right)$$

$$=\frac{(\boldsymbol{a})_{-\mu}(\boldsymbol{c})_{\mu}}{(\boldsymbol{b})_{-\mu}(\boldsymbol{d})_{\mu}}\sum_{m=0}^{\infty}\frac{(\lambda+2m)(-\mu)_m}{m!(\lambda+m)_{\mu+1}}B_m(t)$$

$$\times F^{1+r+p_0:p_1;\dots;p_n}_{2+s+q_0:q_1;\dots;q_n}\left(\begin{matrix}\mu+1,\ \boldsymbol{c}+\mu,\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\dots;\boldsymbol{a}_n;\\ \mu-m+1,\ \lambda+\mu+m+1,\ \boldsymbol{d}+\mu,\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\dots;\boldsymbol{b}_n;\end{matrix}\ x_1,\dots,x_n\right) \tag{16}$$

for $p+s+1=q+r$ (and $0<t\leqslant 1$);

$$t^{\mu}F^{p+p_0:p_1;\dots;p_n}_{q+q_0:q_1;\dots;q_n}\begin{pmatrix}x_1t\\ \vdots\\ x_nt\end{pmatrix}$$

$$=\frac{(\boldsymbol{a})_{-\mu}(\boldsymbol{c})_{\mu}}{(\boldsymbol{b})_{-\mu}(\boldsymbol{d})_{\mu}}\sum_{m=0}^{\infty}\frac{(-\mu)_m}{m!}B^*_m(t)$$

$$\times F^{1+r+p_0:p_1;\dots;p_n}_{1+s+q_0:q_1;\dots;q_n}\left(\begin{matrix}\mu+1,\ \boldsymbol{c}+\mu,\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\dots;\boldsymbol{a}_n;\\ \mu-m+1,\ \boldsymbol{d}+\mu,\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\dots;\boldsymbol{b}_n;\end{matrix}\ x_1,\dots,x_n\right) \tag{17}$$

for $p+s+1=q+r$ (and $0<t<\infty$);

$$t^{\mu}F^{p+p_0:p_1;\dots;p_n}_{q+q_0:q_1;\dots;q_n}\begin{pmatrix}x_1t\\ \vdots\\ x_nt\end{pmatrix}$$

$$=\frac{\beta(\boldsymbol{a})_{-\mu}(\boldsymbol{c})_{\mu}}{(\boldsymbol{b})_{-\mu}(\boldsymbol{d})_{\mu}}\sum_{m=0}^{\infty}\frac{(\beta-\alpha m+1)_{\mu-1}(-\mu)_m}{m!}B^{(\alpha)}_m(t)$$

$$\times F^{2+r+p_0:p_1;\dots;p_n}_{1+s+q_0:q_1;\dots;q_n}\left(\begin{matrix}\mu+1,\ \mu-\alpha m+\beta,\ \boldsymbol{c}+\mu,\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\dots;\boldsymbol{a}_n;\\ \mu-m+1,\ \boldsymbol{d}+\mu,\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\dots;\boldsymbol{b}_n;\end{matrix}\ x_1,\dots,x_n\right) \tag{18}$$

for $p+s+1=q+r$ (and $0<t<\infty$); it being understood in every case that

$$1+q_0+q_k-p_0-p_k\geqslant p-q \qquad k=1,\dots,n \tag{19}$$

where the equality holds true when the variables $t$ and $x_1,\dots,x_n$ are appropriately constrained in accordance with (9) and (10). Furthermore, exceptional parameter values which would render either side invalid or undefined are tacitly excluded. Thus, for example, the expansion formula (16) remains valid also for $t=0$, provided that $\mathrm{Re}(\mu)>0$; on the other hand, when $\mu$ in any of these expansion formulae takes on a non-negative integer value $N$, the right-hand side will have to be modified by a suitable limit process in order to (tacitly) avoid division by zero for the summation index (cf Srivastava and Panda 1976, p 142):

$$m=N,\ N+1,\ N+2,\dots \qquad N=0,1,2,\dots. \tag{20}$$

Each of our expansion formulae (16)-(18), and their various specialised or limiting cases, can be applied to derive scores of linearisation relations analogous to (3) and (4) for the products of several generalised hypergeometric polynomials of the types defined by (12)-(14). In particular, for the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the expansion formula (16) with

$$\lambda=\alpha+\beta+1 \qquad p=q=p_0=q_0=r-1=s=0 \qquad c^1=\alpha+1$$
$$p_k-1=q_k=1 \qquad a_k^1=-m_k \qquad a_k^2=\alpha_k+\beta_k+m_k+1$$
$$b_k^1=\alpha_k+1 \qquad k=1,\ldots,n$$

yields the following linearisation relation of the Clebsch-Gordan type (3):

$$\begin{aligned} t^\mu P_{m_1}^{(\alpha_1,\beta_1)}&(1-2x_1t)\ldots P_{m_n}^{(\alpha_n,\beta_n)}(1-2x_nt)\\ &=(\alpha+1)_\mu\binom{\alpha_1+m_1}{m_1}\ldots\binom{\alpha_n+m_n}{m_n}\\ &\quad\times\sum_{m=0}^{\infty}\frac{(\alpha+\beta+2m+1)(-\mu)_m}{(\alpha+1)_m(\alpha+\beta+m+1)_{\mu+1}}P_m^{(\alpha,\beta)}(1-2t)\\ &\quad\times F_{2:1;\ldots;1}^{2:2;\ldots;2}\left(\begin{matrix}\mu+1,\alpha+\mu+1: -m_1,\alpha_1+\beta_1+m_1+1;\ldots;\\ \mu-m+1,\alpha+\beta+\mu+m+2: \alpha_1+1;\ldots;\end{matrix}\right.\\ &\qquad\qquad\left.\begin{matrix}-m_n,\alpha_n+\beta_n+m_n+1;\\ \alpha_n+1;\end{matrix}\,x_1,\ldots,x_n\right). \end{aligned} \tag{21}$$

In a similar manner, if we apply the definition of the classical Laguerre polynomials $L_n^{(\alpha)}(x)$, the expansion formula (17) with

$$p=q=p_0=q_0=r-1=s=0 \qquad c^1=\alpha+1$$
$$p_k=q_k=1 \qquad a_k^1=-m_k \qquad b_k^1=\alpha_k+1 \qquad k=1,\ldots,n$$

would reduce to the linearisation relation:

$$\begin{aligned} t^\mu L_{m_1}^{(\alpha_1)}&(x_1t)\ldots L_{m_n}^{(\alpha_n)}(x_nt)\\ &=(\alpha+1)_\mu\binom{\alpha_1+m_1}{m_1}\ldots\binom{\alpha_n+m_n}{m_n}\sum_{m=0}^{\infty}\frac{(-\mu)_m}{(\alpha+1)_m}L_m^{(\alpha)}(t)\\ &\quad\times F_{1:1;\ldots;1}^{2:1;\ldots;1}\left(\begin{matrix}\mu+1,\alpha+\mu+1: -m_1;\ldots; -m_n;\\ \mu-m+1: \alpha_1+1;\ldots;\alpha_n+1;\end{matrix}\,x_1,\ldots,x_n\right) \end{aligned} \tag{22}$$

which is also of the Clebsch-Gordan type (3).

The multivariable hypergeometric polynomials involved in the coefficients of each of the (Clebsch-Gordan type) linearisation relations (21) and (22) can easily be rewritten in descending powers of $x_1,\ldots,x_n$. Notice also that, in view of a familiar limit relationship (cf Szegö 1975, p 103, equation (5.3.4)), the linearisation relation (22) would follow directly from (21) when we replace $t$ by $t/\beta$, and $x_k$ by $\beta x_k/\beta_k$ $(k=1,\ldots,n)$ and let $\beta,\beta_1,\ldots,\beta_n\to\infty$. More importantly, since

$${}_2F_1\left[\begin{matrix}\alpha+\mu+M+1,-m;\\ \alpha+1;\end{matrix}\,1\right]=\frac{(-\mu)_m(\mu+1)_M}{(\alpha+1)_m(\mu-m+1)_M} \qquad m, M=0,1,2,\ldots \tag{23}$$

by the Chu-Vandermonde theorem, which is a well known special case of the Gaussian summation theorem (cf Srivastava and Karlsson 1985, p 19, equation 1.2(20)), it is not difficult to rewrite the linearisation relation (22) in the elegant form:

$$t^{\mu}L_{m_1}^{(\alpha_1)}(x_1 t)\ldots L_{m_n}^{(\alpha_n)}(x_n t)=\sum_{m=0}^{\infty}\gamma_m(\mu;x_1,\ldots,x_n)L_m^{(\alpha)}(t) \tag{24}$$

where, for convenience,

$$\begin{aligned}\gamma_m(\mu;x_1,\ldots,x_n)=(\alpha+1)_{\mu}\binom{\alpha_1+m_1}{m_1}\ldots\binom{\alpha_n+m_n}{m_n}\\ \times F_A^{(n+1)}[\alpha+\mu+1,-m_1,\ldots,-m_n,-m;\\ \alpha_1+1,\ldots,\alpha_n+1,\alpha+1;x_1,\ldots,x_n,1]\end{aligned} \tag{25}$$

in terms of one of Lauricella's hypergeometric functions of $n+1$ variables (e.g. Srivastava and Karlsson 1985, p 33, equation 1.4(1)).

The linearisation relation (22) corresponding to the *restricted* product in (5) was given by Niukkanen (1985). On the other hand, the equivalent expansion (24) with $\mu=0$ immediately yields the following result due to Erdélyi (1938):

$$\begin{aligned}L_{m_1}^{(\alpha_1)}(x_1 t)\ldots L_{m_n}^{(\alpha_n)}(x_n t)\\ =\binom{\alpha_1+m_1}{m_1}\ldots\binom{\alpha_n+m_n}{m_n}\sum_{m=0}^{m_1+\ldots+m_n}L_m^{(\alpha)}(t)\\ \times F_A^{(n+1)}[\alpha+1,-m_1,\ldots,-m_n,-m;\\ \alpha_1+1,\ldots,\alpha_n+1,\alpha+1;x_1,\ldots,x_n,1]\end{aligned} \tag{26}$$

which has since been reproduced (with proper credits) in numerous subsequent works.

We should like to mention here that Erdélyi's linearisation relation (26) can be extended fairly easily to an expansion (or multiplication) theorem for a class of multivariable hypergeometric polynomials in the form:

$$\begin{aligned}F_{q:\ q_1;\ldots;\ q_n}^{p:1+p_1;\ldots;1+p_n}\begin{pmatrix}\boldsymbol{a}:-m_1,\boldsymbol{a}_1;\ldots;-m_n,\boldsymbol{a}_n; & \\ \boldsymbol{b}:\quad \boldsymbol{b}_1;\ldots;\quad \boldsymbol{b}_n; & x_1 t,\ldots,x_n t\end{pmatrix}\\ =\sum_{m=0}^{m_1+\ldots+m_n}\binom{\alpha+m-1}{m}{}_{1+p}F_{1+q}\left[\begin{matrix}-m,\boldsymbol{a};\\ \alpha,\boldsymbol{b};\end{matrix}\ t\right]\\ \times F_{0:\ q_1;\ldots;\ q_n;1}^{1:1+p_1;\ldots;1+p_n;1}\begin{pmatrix}\alpha:-m_1,\boldsymbol{a}_1;\ldots;-m_n,\boldsymbol{a}_n;-m; & \\ \text{—}:\quad \boldsymbol{b}_1;\ldots;\quad \boldsymbol{b}_n;\ \alpha; & x_1,\ldots,x_n,1\end{pmatrix}\end{aligned} \tag{27}$$

which is derivable from a more general result involving the generalised Lauricella function (cf Srivastava 1971, p 114; see also Srivastava and Manocha 1984, p 262, problem 5). As a matter of fact, in view of the hypergeometric identity (23), this last result (27) corresponds to an obvious (terminating) version of the special case $\mu=0$ of the following consequence of our expansion formula (17):

$$\begin{aligned}t^{\mu}F_{q:q_1;\ldots;q_n}^{p:p_1;\ldots;p_n}\begin{pmatrix}\boldsymbol{a}:\boldsymbol{a}_1;\ldots;\boldsymbol{a}_n; & \\ \boldsymbol{b}:\boldsymbol{b}_1;\ldots;\boldsymbol{b}_n; & x_1 t,\ldots,x_n t\end{pmatrix}\\ =\frac{(\alpha)_{\mu}(\boldsymbol{a})_{-\mu}}{(\boldsymbol{b})_{-\mu}}\sum_{m=0}^{\infty}\binom{\alpha+m-1}{m}{}_{1+p}F_{1+q}\left[\begin{matrix}-m,\boldsymbol{a}-\mu;\\ \alpha,\boldsymbol{b}-\mu;\end{matrix}\ t\right]\\ \times F_{0:q_1;\ldots;q_n;1}^{1:p_1;\ldots;p_n;1}\begin{pmatrix}\alpha+\mu:\boldsymbol{a}_1;\ldots;\boldsymbol{a}_n;-m; & \\ \text{—}:\boldsymbol{b}_1;\ldots;\boldsymbol{b}_n;\ \alpha; & x_1,\ldots,x_n,1\end{pmatrix}\end{aligned} \tag{28}$$

which evidently holds true for an essentially arbitrary parameter $\mu$.

## 3. Concluding remarks and observations

We should like to remark that the literature contains a wide variety of addition theorems for various special functions (e.g. Srivastava *et al* 1983). In the case of the classical Laguerre polynomials, a few of the available addition theorems were discussed by Niukkanen (1985). A particularly elegant result for these polynomials is the following addition theorem of Srivastava (1972, p 6, equation (10)):

$$
\begin{aligned}
L_m^{(\alpha)}(x)L_m^{(\beta)}(y)
&= \binom{\alpha+m}{m}\binom{\beta+m}{m}\binom{\gamma+m}{m}^{-1} \\
&\quad \times \sum_{r,s=0}^{r+s\leqslant m} \frac{(\gamma+1)_r(\beta-\gamma)_s x^r y^{r+s}}{r!\,s!\,(\alpha+1)_r(\beta+1)_{r+s}}\, \xi_{rs} L_{m-r-s}^{(\gamma+2r+s)}(x+y)
\end{aligned}
\tag{29}
$$

where, for convenience,

$$
\xi_{rs} = {}_3F_2\left[\begin{matrix} -s,\ \alpha-\gamma,\ -\beta-r-s; \\ \alpha+r+1,\ \gamma-\beta-s+1; \end{matrix}\ -x/y\right].
\tag{30}
$$

It may be of interest to observe from (30) that $\xi_{rs}=1$ when $\gamma=\alpha$, and thus (29) reduces immediately to the significantly simpler form:

$$
L_m^{(\alpha)}(x)L_m^{(\beta)}(y) = \binom{\beta+m}{m} \sum_{r,s=0}^{r+s\leqslant m} \frac{(\beta-\alpha)_s x^r y^{r+s}}{r!\,s!\,(\beta+1)_{r+s}}\, L_{m-r-s}^{(\alpha+2r+s)}(x+y)
\tag{31}
$$

whose special case when $\beta=\alpha$ would yield one of several such addition theorems considered extensively by Bailey (1936, p 219, equation (5.4); 1939, p 60, equation (1.1)).

By Kummer's first formula (cf Srivastava and Karlsson 1985, p 322, equation 9.4(183)), the standard confluent hypergeometric ${}_1F_1$ representation for the Laguerre polynomials $L_n^{(\alpha)}(x)$ may be rewritten at once as

$$
L_n^{(\alpha)}(x) = \binom{\alpha+n}{n} \exp(x)\, {}_1F_1\left[\begin{matrix} \alpha+n+1; \\ \alpha+1; \end{matrix}\ -x\right].
\tag{32}
$$

Now make use of (32) in an expansion formula recorded already by Srivastava (1985a, p L230, equation (20)) which indeed follows readily from a more general result due to Srivastava and Daoust (1969, p 456, equation (4.3)). We thus obtain the following linearisation relation for the Laguerre polynomials:

$$
\begin{aligned}
L_{m_1}^{(\alpha_1)}(x_1 t) \ldots L_{m_n}^{(\alpha_n)}(x_n t)
&= \binom{\alpha_1+m_1}{m_1} \cdots \binom{\alpha_n+m_n}{m_n} \exp[-(x-x_1-\ldots-x_n)t] \\
&= \sum_{m=0}^{s+M} \frac{\alpha+2m}{\alpha+m} \binom{\alpha+s+M}{s+M-m}^{-1} \frac{(xt)^m}{m!} L_{s+M-m}^{(\alpha+2m)}(xt) \\
&\quad \times F_{1:1;\ldots;1}^{2:1;\ldots;1}\left(\begin{matrix} -m,\ \alpha+m\colon & \alpha_1+m_1+1;\ \ldots;\ & \alpha_n+m_n+1; \\ \alpha+s+M+1\colon & \alpha_1+1;\ \ldots;\ & \alpha_n+1; \end{matrix}\ x_1/x, \ldots, x_n/x\right)
\end{aligned}
\tag{33}
$$

where, for convenience,

$$
M = m_1+\ldots+m_n \qquad s = \alpha_1+\ldots+\alpha_n-\alpha+1
$$

$\alpha$ being so constrained that $s$ is a non-negative integer. Formula (33) with $n = 2$ would provide the *corrected* (and modified) version of a result proved, in a markedly different and involved manner, by Niukkanen (1985, p 1413, equation (48)). In fact, Niukkanen's error can be traced back to the missing factor $(\frac{1}{2}z)^{\gamma-\alpha_1-\alpha_2}$ on the left-hand side of a well known result reproduced and used incorrectly by him (cf Niukkanen 1985, p 1412, equation (47)).

Formula (33) with

$$x = x_1 + \ldots + x_n$$

would immediately yield a class of addition theorems for Laguerre polynomials, which (for $n = 2$) would correspond essentially to the corrected version of another known result (Niukkanen 1985, p 1414, equation (52)).

A number of further applications of each one of the expansion (or multiplication) formulae and linearisation relations presented in this paper to various other families of orthogonal polynomials (or to simpler special functions of one and more variables) can indeed be given in a manner outlined above fairly adequately. Moreover, these multivariable polynomial expansions are also capable of yielding various desired linearisation relations of the *modified* Clebsch–Gordan type (4) for each of the classical orthogonal polynomials named in § 1, as well as for numerous other hypergeometric polynomials of the types considered in this paper.

## Acknowledgments

The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under grant no A-7353.

## References

Bailey W N 1936 *Proc. Lond. Math. Soc.* **41** 215–20
—— 1939 *Q. J. Math.* **10** 60–6
Erdélyi A 1938 *J. Lond. Math. Soc.* **13** 154–6
Niukkanen A W 1983 *J. Phys. A: Math. Gen.* **16** 1813–25
—— 1984 *J. Phys. A: Math. Gen.* **17** L731–6
—— 1985 *J. Phys. A: Math. Gen.* **18** 1399–417
Srivastava H M 1971 *Bull. Soc. Math. Grèce* **11** 66–70
—— 1972 *Boll. Un. Mat. Ital.* **5** 1–6
—— 1985a *J. Phys. A: Math. Gen.* **18** L227–34
—— 1985b *J. Phys. A: Math. Gen.* **18** 3079–85
—— 1987 *J. Phys. A: Math. Gen.* **20** 847–55
Srivastava H M and Daoust M C 1969 *Neder. Akad. Wetensch. Indag. Math.* **31** 449–57
Srivastava H M and Karlsson P W 1985 *Multiple Gaussian Hypergeometric Series* (New York: Halsted/Wiley)
Srivastava H M, Lavoie J-L and Tremblay R 1983 *Can. Math. Bull.* **26** 438–45
Srivastava H M and Manocha H L 1984 *A Treatise on Generating Functions* (New York: Halsted/Wiley)
Srivastava H M and Panda R 1974 *Comm. Math. Univ. St Paul* **23**(1) 7–14
—— 1976 *J. Reine Angew. Math.* **288** 129–45
Szegö G 1975 *Orthogonal Polynomials* (Providence, RI: American Mathematical Society)